Scientific Annals of Computer Science vol. 36 (1), 2026, pp. 175–207
doi: 10.47743/SACS.2026.1.175

The 'Ancestor' of Mundici's Equivalence

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Abstract

This article provides an exposition of the main results presented in our book "Four Functors: the 'ancestor' of Mundici's equivalence between unital commutative l-groups and MV algebras" appeared in autumn of 2025.

Daniele Mundici has proved, in 1986, that the category of *unital commutative l-groups* is equivalent to the category of *MV algebras*, by proving, in one step, the existence of the two quasi-inverse functors $\mathbf{\Gamma}$ and $\mathbf{\Xi}$.

Marco Abbadini has presented in 2021 his equivalence á la Mundici for commutative l-monoids, in two steps, with four functors. Mundici's equivalence itself can be proved in two steps, with four functors, by introducing the intermediary category of *positive-unital commutative l-groups*.

We have presented in our book the following 'ancestor' of Mundici's equivalence, by Abbadini's 'style' (i.e. in two steps, with four functors): The category of strong m_0-ME structures is not equivalent to the (intermediary) category of strong m_0-ME positive cones, but the category of strong m_0-ME positive cones is equivalent to the category of XY algebras.

Keywords: MV algebra, XY algebra, unital commutative l-group, strong m_0-ME structure, strong m_0-ME positive cone
AMS classification (2020): 06D35, 06A75, 03G99

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1 Introduction

MV algebras were introduced in 1958, by C.C. Chang [4], in order to prove the completeness of infinite-valued Łukasiewicz logic. The Chang's definition of MV algebras presents them as algebras $(A, \oplus, \cdot, ^-, 0, 1)$ with 20 axioms, while our days definition presents them as algebras $(A, \oplus, ^-, 0)$ with only 6 axioms (see Definition 2.6). The reader has at least [5] and [26] to find more about MV algebras. Equivalences of MV algebras are known in the literature under several names: Lacava's *Ł-algebras* [20], Rodríguez's *Wajsberg algebras* [25, 7], Bosbach's *bricks* [2], Komori's *CN-algebras* [19], Buff's *S-algebras* [3], *bounded commutative BCK algebras* [24] - where the BCK algebras were introduced in 1966 by K. Iseki [17] (see also [18, 10, 13]). In the surveyed book, [15], two new axiomatizations of MV algebras are presented. An important result in MV algebras is *Chang's subdirect representation theorem* ([5], Th. 1.3.3), stating that an equation is valid in any MV algebra if and only if it is valid in any totally ordered MV algebra (MV chain); the proof of this theorem uses *Zorn's lemma*, which is equivalent to the *axiom of choice*.

Daniele Mundici has proved in 1986 [23, 5] that the category of *lattice-ordered commutative groups with strong unit* (or unital commutative l-groups, for short) is equivalent to the category of *MV algebras*, by proving the existence of the two quasi-inverse functors $\mathbf{\Gamma}$ and $\mathbf{\Xi}$ ([5], Cor. 7.1.8). He uses the axiom of choice (by using Chang's subdirect representation theorem) in his proof and makes the proof in one step, namely:

- In [5], Chapter 2, it is proved that, for any MV algebra A, there exists a unital commutative l-group that envelops A. First, a lattice ordered monoid M_A of good sequences of A is constructed from A; then, a unital commutative l-group G_A is constructed from M_A (in a way that is analogous to the definition of $\mathbb{Z}$ from $\mathbb{N}$).

- Conversely, in [5], Chapter 2 also, it is proved that, for any commutative l-group G with strong unit u, there exists an MV algebra, $\Gamma(G, u)$, that envelops G (see [5], Def. 2.1.1): let $(G, \vee, \wedge, +, -, 0)$ be an l-group $(x \leq y \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x)$. For any element $u \in G$, $u > 0$ (not necessarily u being a strong unit of G), let $[0, u] \stackrel{def.}{=} \{x \in G \mid 0 \leq x \leq u\}$ and, for each $x, y \in [0, u]$, define: $x \oplus y \stackrel{def.}{=} u \wedge (x + y)$, $\neg x \stackrel{def.}{=} u - x$. The structure $([0, u], \oplus, \neg, 0)$ is denoted $\Gamma(G, u)$. It is proved ([5], Prop. 2.1.2) that $\Gamma(G, u)$ is an MV algebra. By [5], Prop. 2.1.5, $\mathbf{\Gamma}$ is a functor. The functor $\mathbf{\Gamma}$ defines a natural equivalence between the

category of commutative l-groups with strong unit and the category of MV algebras ([5], Cor. 7.1.8).

Marco Abbadini has presented in 2021 [1] his equivalence á la Mundici for commutative l-monoids, in two steps, with four functors: The category of *unital lattice-ordered commutative monoids* (or unital commutative l-monoids, for short) (roughly speaking, unital commutative l-groups without the unary operation $x \longmapsto -x$) is equivalent to the category of *MV-monoidal algebras* (roughly speaking, MV algebras without the unary operation $x \longmapsto x^-$), by proving the existence of the quasi-inverse functors $\mathbf{\Gamma}_\star$ and $\mathbf{\Xi}_\star$. He does not use the axiom of choice and makes the proof in two steps, with four functors, i.e. he introduces the intermediary category of *positive-unital commutative l-monoids* and four intermediary functors: the quasi-inverse functors $\mathbf{c}^+_\star$ and $\mathbf{T}_\star$, in step 1, and the quasi-inverse functors $\mathbf{G}^+_\star$ and $\mathbf{U}_\star$, in step 2, such that $\mathbf{\Gamma}_\star = \mathbf{U}_\star \circ \mathbf{c}^+_\star$ and $\mathbf{\Xi}_\star = \mathbf{T}_\star \circ \mathbf{G}^+_\star$. He obtains the original Mundici's equivalence as a corollary of his equivalence in [1], Appendix A.

Recall ([21], Chapter IV, Section 4 - conform [1]) that:

> Two functors $\mathbf{F} : \mathcal{C}(\mathbf{X}) \longrightarrow \mathcal{C}(\mathbf{Y})$ and $\mathbf{G} : \mathcal{C}(\mathbf{Y}) \longrightarrow \mathcal{C}(\mathbf{X})$ are called *quasi-inverse*, if the two functors $\mathbf{G} \circ \mathbf{F} : \mathcal{C}(\mathbf{X}) \longrightarrow \mathcal{C}(\mathbf{X})$ and $\mathbf{F} \circ \mathbf{G} : \mathcal{C}(\mathbf{Y}) \longrightarrow \mathcal{C}(\mathbf{Y})$ are naturally isomorphic to the identity functors on $\mathcal{C}(\mathbf{X})$ and $\mathcal{C}(\mathbf{Y})$, respectively. Two categories $\mathcal{C}(\mathbf{X})$ and $\mathcal{C}(\mathbf{Y})$ are *equivalent*, if there exists two quasi-inverse functors $\mathbf{F} : \mathcal{C}(\mathbf{X}) \longrightarrow \mathcal{C}(\mathbf{Y})$ and $\mathbf{G} : \mathcal{C}(\mathbf{Y}) \longrightarrow \mathcal{C}(\mathbf{X})$.

Mundici has made the proof in one step, with two functors, but the proof can be made in two steps, with four functors - as Abbadini did - and it is sketched in the Introduction of our book - by introducing the intermediary category of *positive-unital commutative l-groups* and four intermediary functors: $\mathbf{c}^+$ and $\mathbf{T}$, in step 1, and $\mathbf{G}^+$ and $\mathbf{U}$, in step 2, such that $\mathbf{\Gamma} = \mathbf{U} \circ \mathbf{c}^+$ and $\mathbf{\Xi} = \mathbf{T} \circ \mathbf{G}^+$; thus, the category of unital commutative l-groups is equivalent to the category of positive-unital commutative l-groups (step 1) and the category of positive-unital commutative l-groups is equivalent to the category of MV algebras (step 2).

There existed an *old framework of commutative algebras of logic*, i.e. of algebras with an implication $\to$ and a constant 1 (representing the 'truth') (and a negation $^-$, $x^- \stackrel{def.}{=} x \to 0$, when there is 0 s.t. $0 \leq x$ for all x), as basic operations, framework having in its 'center' the *BCK algebra*, begun

in papers [11]. It was completed in the monograph [16], where the *M, ME, ML* and *MEL algebras* were introduced (in the non-commutative case); the Waisberg algebras, the implicative-Boolean algebras and the commutative implicative-groups belong to this framework too. A *(left-) MEL algebra* is an algebra $(A, \to, 1)$ of type $(2, 0)$ verifying the axioms (M) $(1 \to x = x)$, (Ex) $(x \to (y \to z) = y \to (x \to z))$ and (La) $(x \to 1 = 1)$, where the binary relation $x \leq y \overset{def.}{\iff} x \to y = 1$ can be defined. Then, searching around the *(left-) MV algebra* - an algebra $(A, \odot, ^-, 1)$ verifying the axioms (PU) $(1 \odot x = x)$, (Pcomm) $(x \odot y = y \odot x)$, (Pass) $(x \odot (y \odot z) = (x \odot y) \odot z)$, $x \odot 0 = 0$ $(0 \overset{def.}{=} 1^-)$, (DN) and ($\wedge_m$-comm) $((x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x)$, where the binary relation $x \leq_m y \overset{def.}{\iff} x \odot y^- = 0$ can be defined - we have created, in [12] and then in the monograph [13], an analogous *framework of commutative algebras having in its 'center' the m-BCK algebras* ('m' coming from 'magma'), containing the new created m-MEL algebras, m-BCK algebras etc. and also the MV algebras, the Boolean algebras and the commutative groups. A *(left-) m-MEL algebra* is an algebra $(A, \odot, ^-, 1)$ with $0 \overset{def.}{=} 1^-$ verifying (PU), (Pcomm), (Pass) and (m-La) $(x \odot 0 = 0)$, where the binary relation $\leq_m$ can be defined. A (left-) MV algebra is an involutive (left-) m-MEL algebra verifying ($\wedge_m$-comm); it is a particular case of m-BCK algebra. The framework centered on BCK algebras and the framework centered on m-BCK algebras are connected by 'bridge' theorems in the involutive case, i.e. when (DN) $(x^= = x)$ holds; for examples, $x \leq y \iff x \leq_m y$, (Ex) $\iff$ (Pass), (La) $\iff$ (m-La), involutive MEL algebras are definitionally equivalent (d.e.) with involutive m-MEL algebras ([13], Theorem 17.1.2), involutive BCK algebras are d.e. with (involutive) m-BCK algebras ([13], Corollary 17.1.4), Wajsberg algebras are d.e. with MV algebras [7] ([13], Corollary 17.1.5), implicative-Boolean algebras are d.e. with Boolean algebras ([13], Corollary 17.1.7); the commutative implicative-groups are d.e. with the commutative groups ([13], Theorem 16.2.5 or 17.2.1).

Thus, we have had the background material to generalize Mundici's equivalence, to find its 'ancestor' and, hence, to write the surveyed book, [15]. The role of *unital commutative l-groups* is played by the *strong m_0-ME structures*, which are involutive m_0-ME algebras verifying some properties; the intermediary role of *positive-unital commutative l-groups* is played by the *strong m_0-ME positive cones*, which are involutive m_0-ME algebras too verifying some properties; the role of *MV algebras* is played by the *XY algebras*, which are involutive m-MEL algebras verifying the properties

(XX) and (YY). The *strong m_0-ME structures*, the *strong m_0-ME positive cones* and the *XY algebras* are new notions belonging to the new framework centered on m-BCK algebras. We do not use the axiom of choice in our proof (in fact, it is the only choice in our case, since there are no proper XY chains - the XY chains are in fact MV chains). We do not follow the proof of Mundici's equivalence from [5], in one step, with two functors, but the proof of Abbadini's equivalence from [1], in two steps, with four functors (in fact, it is the only choice in our case too, since the corresponding functors $\mathbf{\Gamma_m}$ and $\mathbf{\Xi_m}$ are not quasi-inverse, but the intermediary two functors $\mathbf{G_m^+}$ and $\mathbf{U_m}$ - of step 2 - are quasi-inverse). So, we establish the following generalization of Mundici's equivalence ([15], Th. 10.1.1): The category of strong m_0-ME structures is not equivalent to the category of strong m_0-ME positive cones, but the category of strong m_0-ME positive cones is equivalent to the category of XY algebras. We show how the original Mundici's equivalence is a particular case of our results (see [15], Cor. 2.2.68, and also Connections from [15], Chapter 3 and **Appendix C**).

The book [15] is organized in ten chapters and four parts, as follows: Part 1 (The categories involved) contains chapters 2 (XY algebras) and 3 (Strong m_0-ME structures and positive cones); Part 2 (The first step: the functors $\mathbf{c_m^+}$ and $\mathbf{T_m}$) contains chapters 4 (The functor $\mathbf{c_m^+}$), 5 (The functor $\mathbf{T_m}$) and 6 (Connections between $\mathbf{c_m^+}$ and $\mathbf{T_m}$); Part 3 (The second step: the functors $\mathbf{G_m^+}$ and $\mathbf{U_m}$) contains chapters 7 (The functor $\mathbf{G_m^+}$), 8 (The unit interval functor $\mathbf{U_m}$) and 9 (Connections between $\mathbf{G_m^+}$ and $\mathbf{U_m}$); Part 4 (Connections between the four functors) contains Chapter 10 (Final connections) and an **Appendix**, as Chapter 11, containing: in **Appendix A**, three Figures from [13], [14] connecting the classes of algebras involved in the book; in **Appendix B**, a long proof from Chapter 2; in **Appendix C**, some long proofs from Chapter 3. The book has 317 pages.

As the readers will see, the strong m_0-ME structures, the strong m_0-ME positive cones and the XY algebras - the 'ancestors' of unital commutative l-groups, positive-unital commutative l-groups and MV algebras, respectively - are not looking very nice, some are even 'ugli'. But, they are - as all ancestors are - very important, because they help us to better understand their descendants. 'Digging' for these 'ancestors' was a very difficult work, work most probably impossible without the help of Prover9-Mace4 tool given to us by William W. McCune (1953-2011) [22]. Abbadini's results and the wonderful instrument which is Prover9-Mace4 (version Dec-2007 on Windows) were the two major helps in writing the book.

We due many thanks to George Georgescu, for sending us Abbadini's paper [1] on October 22, 2024, and to Michael Kinyon, for his friendly help with some proofs by his more powerfull PROVER9-MACE4 on Linux (when our PROVER9 failed to provide a proof, we used to ask Michael to try with his PROVER9).

This survey paper is organized in five sections: Section 2 (XY algebras), Section 3 (Strong m_0-ME structures. Strong m_0-ME positive cones), Section 4 (The two steps and the four functors) and Section 5 (Future work).

2 XY Algebras

In this section, we recall, from the framework centered on m-BCK algebras, the involutive m-MEL algebras and the MV algebras. The two new axiomatizations of MV algebras from [15] are presented.

The ancestor (generalization) of MV algebra, called *XY algebra*, belongs to this framework too.

2.1 Involutive m-MEL Algebras

Definitions 2.1 ([15], Defs. 2.1.1) (See [13], subsection 6.1.1)

(1) A right-m-MEL algebra is an algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ *of type* $(2, 1, 0)$ *verifying the following axioms: for all* $x, y, z \in A$,

(SU) $x \oplus 0 = x = 0 \oplus x$,
(Scomm) $x \oplus y = y \oplus x$,
(Sass) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(Neg0-1) $1 \stackrel{def.}{=} 0^-$,
(Neg1-0) $1^- = 0$,
(m-La^R) $x \oplus 1 = 1$.

(1') Dually, a left-m-MEL algebra is an algebra $\mathcal{A} = (A, \odot, ^-, 1)$ *of type* $(2, 1, 0)$ *verifying the following axioms: for all* $x, y, z \in A$,

(PU) $x \odot 1 = x = 1 \odot x$,
(Pcomm) $x \odot y = y \odot x$,
(Pass) $(x \odot y) \odot z = x \odot (y \odot z)$,
(Neg1-0) $0 \stackrel{def.}{=} 1^-$,
(Neg0-1) $0^- = 1$,
(m-La) $x \odot 0 = 0$.

Note that we have given names to the dual algebras [10], [16], [12]: "left" algebra and "right" algebra, names related to the left-continuity of a t-norm $\odot$ and to the right-continuity of a t-conorm $\oplus$, respectively. Note also that we have worked with left-algebras in [12], [13], [14], since the *truth* in logic is represented by 1.

Denote by $\mathbf{m\text{-}MEL}^R$ the class of all right-m-MEL algebras and by **m-MEL** the class of all left-m-MEL algebras.

Definitions 2.2 ([15], Defs. 2.1.2) ([13], Def. 6.4.1)

(1) An involutive right-m-MEL algebra is a right-m-MEL algebra $\mathcal{A}$ verifying the additional axiom: for each $x \in A$,

(DN) $(x^-)^- = x$ *(or, better, $x^= = x$).*

(1') An involutive left-m-MEL algebra is a left-m-MEL algebra $\mathcal{A}$ verifying the additional axiom (DN).

Denote by $\mathbf{m\text{-}MEL}^R_{(\mathrm{DN})}$ the class of all involutive right-m-MEL algebras and by $\mathbf{m\text{-}MEL}_{(\mathrm{DN})}$ the class of all involutive left-m-MEL algebras.

Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. Let us introduce a new operation, $\odot$, defined by ([13], (6.5)): for all $x, y \in A$,

$$x \odot y \stackrel{def.}{=} (x^- \oplus y^-)^-. \tag{1}$$

Then, we have the following result.

Proposition 2.3 ([15], Prop. 2.1.3) (See ([13], Cor. 6.4.3))

Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. Then, the algebra $(A, \odot, ^-, 1)$ is an involutive left-m-MEL algebra.

It follows that $\mathcal{A} = (A, \oplus, ^-, 0)$ verifies (PU), (Pcomm), (Pass), (m-La) too, and

$$x \oplus y = (x^- \odot y^-)^-. \tag{2}$$

2.1.1 Two Binary Relations, $\leq_m$ and $\leq^M_m$, on $\mathcal{A}$

First, let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-m-MEL algebra. We define the following basic internal binary relation (i.e. defined by using the operations on A) ([13], Chapter 6, section 6.1): for all $x, y \in A$,

$$x \leq_m y \stackrel{def.}{\Longleftrightarrow} y \oplus x^- = 1 \ (\Longleftrightarrow x \odot y^- = 0). \tag{3}$$

We shall denote:

- the *reflexivity* of $\leq_m$, by (m-Re^R),
- the *antisymmetry* of $\leq_m$, by (m-An^R) and
- the *transitivity* of $\leq_m$, by (m-Tr^R),

hence we have the following properties:

(m-Re^R) $x \oplus x^- = 1$,
(m-Re'^{R}) $x \leq_m x$;

(m-An^R) $y \oplus x^- = 1$ and $x \oplus y^- = 1$ imply $x = y$,
(m-An'^{R}) $x \leq_m y$ and $y \leq_m x$ imply $x = y$;

(m-Tr^R) $y \oplus x^- = 1$ and $z \oplus y^- = 1$ imply $z \oplus x^- = 1$,
(m-Tr'^{R}) $x \leq_m y$ and $y \leq_m z$ imply $x \leq_m z$.

Note also that the property (m-La^R) ($x \oplus 1 = 1$) means $0 \leq_m x$, denoted by (m-La'^{R}).

Now, let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. We define the following new operations and a new binary relation ([13], Chapter 6, subsection 6.4.3): for all $x, y \in A$,

$$x \vee_m^M y \overset{def.}{=} y \oplus (y \oplus x^-)^-, \quad x \wedge_m^M y \overset{def.}{=} (x^- \vee_m^M y^-)^- = y \odot (y \odot x^-)^- \text{ and}$$

$$x \leq_m^M y \overset{def.}{\iff} y \vee_m^M x = y \iff x \oplus (x \oplus y^-)^- = y. \tag{4}$$

2.1.2 Totally Ordered (Involutive) m-MEL Algebras

We shall introduce now the following two definitions (the dual ones are omitted).

Definitions 2.4 ([15], Defs. 2.1.7)

(1) A right-m-MEL algebra $\mathcal{A}$ is *totally ordered by* $\leq_m$ if, for all $x, y \in A$, we have:

(tot-ord-m) $x \leq_m y$ *or* $y \leq_m x$,

i.e. if we have: $y \oplus x^- = 1$ or $x \oplus y^- = 1$.

(2) An involutive right-m-MEL algebra $\mathcal{A}$ is *totally ordered by* $\leq_m^M$ if, for all $x, y \in A$, we have:

$$\text{(tot-ord-m-M)} \quad x \leq_m^M y \quad \textit{or} \quad y \leq_m^M x,$$

i.e. if we have: $x \oplus (x \oplus y^-)^- = y$ or $y \oplus (y \oplus x^-)^- = x$.

Note that, if a right-m-MEL algebra $\mathcal{A}$ is *totally ordered by* $\leq_m$, then $x \oplus x^- = 1$, for all $x \in A$, i.e. the property of reflexivity, (m-Re^R), holds.

Note also that, if an involutive right-m-MEL algebra $\mathcal{A}$ is *totally ordered by* $\leq_m^M$, then $x \oplus (x \oplus x^-)^- = x$, for all $x \in A$.

2.1.3 Particular m-MEL Algebras

Recall the following definitions from [12], [13].

Definitions 2.5 ([15], Defs. 2.1.10) (The dual ones are omitted)

(1) A right-m-BE algebra is a right-m-MEL algebra verifying additionally the axiom (m-Re^R).

(3) A right-m-pre-BCK algebra is a right-m-BE algebra $\mathcal{A}$ verifying additionally the axiom: for all $x, y, z \in A$,

$$\text{(m-B}^R\text{)} \quad ((x \oplus y^-)^- \oplus (x \oplus z)) \oplus (y \oplus z)^- = 1.$$

(3) A right-m-aBE algebra is a right-m-BE algebra verifying the additional axiom (m-An^R).

(4) A right-m-BCK algebra is a right-m-aBE algebra verifying additionally the above axiom (m-B^R).

Denote by **m-BE**^{R}, **m-pre-BCK**^{R}, **m-aBE**^{R}, **m-BCK**^{R} the classes of all corresponding right-algebras.

Denote by $\mathbf{X}_{(\mathrm{DN})}$ the class of all involutive members (i.e. verifying (DN)) of the class $\mathbf{X}$ of algebras.

2.2 MV Algebras. Two New Axiomatizations

Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra.

Consider the property ($\vee_m$-comm) = (Vcomm$_m^M$) (commutativity of $\vee_m^M$), defined as follows (see for example [13]): for all $x, y \in A$,

$(\vee_m\text{-comm})$ $\quad x \vee_m^M y = y \vee_m^M x.$

We recall the our days definition of an MV algebra and some properties, and we present two new definitions of MV algebras.

Definitions 2.6 ([15], Defs. 2.2.1) ([13], Defs. 6.5.1)

(i) A right-MV algebra is an algebra $\mathcal{A} = (A, \oplus, ^- = {}^{-^R}, 0)$ *verifying: for all* $x, y, z \in A$,

(SU) $\quad 0 \oplus x = x (= x \oplus 0)$,
(Scomm) $\quad x \oplus y = y \oplus x$,
(Sass) $\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(m-La^R) $\quad x \oplus 1 = 1$, *where* $1 \overset{def.}{=} 0^-$ *(i.e.* (Neg0-1) *holds),*
(DN) $\quad (x^-)^- = x$,
$(\vee_m\text{-comm})$ $\quad (x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

(i') Dually, a left-MV algebra is an algebra $\mathcal{A} = (A, \odot, ^- = {}^{-^L}, 1)$ *verifying: for all* $x, y, z \in A$,

(PU) $\quad 1 \odot x = x (= x \odot 1)$,
(Pcomm) $\quad x \odot y = y \odot x$,
(Pass) $\quad x \odot (y \odot z) = (x \odot y) \odot z$,
(m-La) $\quad x \odot 0 = 0$, *where* $0 \overset{def.}{=} 1^-$ *(i.e.* (Neg1-0) *holds),*
(DN) $\quad (x^-)^- = x$,
$(\wedge_m\text{-comm})$ $\quad (x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.

Note that Mundici works with right-MV algebras and, therefore, we have worked in [15] with right-MV algebras too.

Denote by $\mathbf{MV}^R$ the class of all right-MV algebras and by $\mathbf{MV}$ the class of all left-MV algebras. Note that they are varieties.

Note that right-MV algebras verify the property (Neg1-0) too: $1^- \overset{def.}{=} (0^-)^- \overset{(DN)}{=} 0$ and that left-MV algebras verify (Neg0-1) too.

We recall the following important remark, which was the motivation of paper [12] and, hence, of the monograph [13]:

Remark 2.7 ([15], Rem. 2.2.2)

(i) Right-MV algebra is just the involutive right-m-MEL algebra verifying the additional axiom $(\vee_m\text{-comm})$*, i.e.*

$$\mathbf{MV}^R \ = \ \mathbf{m-MEL}^R_{(DN)} \ + \ (\vee_m - \text{comm}).$$

(i') *Left-MV algebra is just the involutive left-m-MEL algebra verifying the additional axiom* ($\wedge_m$-comm)*, i.e.*

$$\mathbf{MV} \;=\; \mathbf{m-MEL}_{(\mathrm{DN})} \;+\; (\wedge_{\mathrm{m}} - \mathrm{comm}).$$

We add the following result.

Proposition 2.8 *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-MV algebra. Then, the algebra $(A, \odot, ^-, 1)$ is a left-MV algebra, where $x \odot y \overset{def.}{=} (x^- \oplus y^-)^-$, by (1), and $1 \overset{def.}{=} 0^-$.*

Proof. Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-MV algebra, i.e. $\mathcal{A}$ is an involutive right-m-MEL algebra verifying ($\vee_m$-comm). Then, by Proposition 2.3, the algebra $(A, \odot, ^-, 1)$ is an involutive left-m-MEL algebra.
We prove now that ($\vee_m$-comm) $\Longrightarrow$ ($\wedge_m$-comm). Indeed,

($\vee_m$-comm) i.e. $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$,
$\Longleftrightarrow$ $(x \odot y^-) \oplus y = (y \odot x^-) \oplus x$, by (2) and (DN),
$\Longleftrightarrow$ $((x \odot y^-)^- \odot y^-)^- = ((y \odot x^-)^- \odot x^-)^-$, by (2),
$\Longrightarrow$ $(x \odot y^-)^- \odot y^- = (y \odot x^-)^- \odot x^-$, by (DN),
$\Longrightarrow$ $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$, for $X := x^-$, $Y := y^-$, by (DN), i.e. ($\wedge_m$-comm) holds.

Thus, the algebra $(A, \odot, ^-, 1)$ is a left-MV algebra. □

Lemma 2.9 ([15], Lem. 2.1.8)
(SU) + (Scomm) + (Sass) + (m-La^R) + (Neg1-0) + (PU)+ (DN) + (tot-ord-m-M) $\Longrightarrow$ ($\vee_m$-comm).

Thus, we obtain the following important result.

Proposition 2.10 ([15], Prop. 2.1.9) *If an involutive right-m-MEL algebra $\mathcal{A}$ is totally ordered by $\leq_m^M$, then it is an MV algebra.*

Proposition 2.11 ([15], Prop. 2.2.3) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-MV algebra. Then, the cancellation property* (C) *holds, where: for all $x, y, z \in A$,*

(C) $\quad z \oplus x = z \oplus y$ *and* $z \odot x = z \odot y$ *imply* $x = y$.

Right-MV algebras verify the properties (m-Re^R) and (m-An^R), by [13], Prop. 6.2.2 (in the dual case), and the properties (m-Tr^R), (m-B^R), by [13], Th. 6.3.5 (in the dual case).

Right-MV algebras are (coincide with) (involutive) m-BCK^R algebras verifying the property ($\vee_m$-comm), by [13], Th. 6.5.10 (in the dual case).

By [13], Prop. 6.4.19 (in the dual case), in any involutive right-m-BE algebra $\mathcal{A} = (A, \oplus, ^-, 0)$, if ($\vee_m$-comm) holds, then $x \leq_m y \Longleftrightarrow x \leq_m^M y$, for all $x, y \in A$.

By [13], Rem. 6.5.9 (in the dual case), in right-MV algebras $\mathcal{A} = (A, \oplus, ^-, 0)$, the two binary relations $\leq_m$ and $\leq_m^M$ are lattice orders with respect to $\vee_m^M$, $\wedge_m^M$ and they coincide, i.e. $\leq_m \Longleftrightarrow \leq_m^M$, the resulting unique lattice order being denoted by $\leq$ (with respect to $\vee = \vee_m^M$, $\wedge = \wedge_m^M$). The lattice $(A, \vee, \wedge)$ is distributive. If the lattice order $\leq$ is totally ordered (or linearly ordered, here), then we say that we have an *MV chain*.

2.2.1 A First New Axiomatization of MV Algebras

Consider the following property:

$(\mathrm{M}_\vee)$ $\quad (x \oplus y) \odot ((x \odot y) \oplus y^-) = x.$

Note that $(\mathrm{M}_\vee)$ is the property (1.15) (for $Y := y^-$, by (DN)) verified by any MV algebra in [5], Prop. 1.6.2.

Corollary 2.12 *([15], Cor. 2.2.6)*
Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. Then,

$$(\vee_m - comm) \iff (M_\vee).$$

Remark 2.13 ([15], Rem. 2.2.7) *The above Corollary 2.12 says that an equivalent definition of MV algebras is the following: A right-MV algebra is an involutive right-m-MEL algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ verifying $(\mathrm{M}_\vee)$, i.e.*

$$\mathbf{MV}^R \ = \ \mathbf{m-MEL}^R_{(\mathrm{DN})} \ + \ (\mathrm{M}_\vee).$$

2.2.2 A Second New Axiomatization of MV Algebras

Consider the following property:

(Vabs_m^M) $\quad x \vee_m^M (x \wedge_m^M y) = x$ (absorbtion of $\vee_m^M$ over $\wedge_m^M$),

where:

$$x \vee_m^M y \overset{def.}{=} y \oplus (y \oplus x^-)^-, \quad x \wedge_m^M y \overset{def.}{=} (x^- \vee_m^M y^-)^- = y \odot (y \odot x^-)^-.$$

Corollary 2.14 ([15], Cor. 2.2.10) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. Then,*

$$(\vee_{\mathrm{m}} - \mathrm{comm}) \iff (\mathrm{Vabs}^{\mathrm{M}}_{\mathrm{m}}).$$

Remark 2.15 ([15], Rem. 2.2.11) *The above Corollary 2.14 says that an equivalent definition of MV algebras is the following: A right-MV algebra is an involutive right-m-MEL algebra* $\mathcal{A} = (A, \oplus, ^-, 0)$ *verifying* (Vabs_m^M), *i.e.*

$$\mathbf{MV}^R \ = \ \mathbf{m-MEL}^R_{(\mathrm{DN})} \ + \ (\mathrm{Vabs}^{\mathrm{M}}_{\mathrm{m}}).$$

2.3 XY Algebras. Cancellative XY Algebras

We shall introduce now a new notion, which will be proved (Cor. 2.30) to be the 'ancestor' of MV algebras.

Definitions 2.16 ([15], Defs. 2.2.13)

(1) A *right-XY algebra* is an involutive right-m-MEL algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ verifying the additional axioms: for all $x, y, z \in A$,
(XX) $(x \oplus y) \odot ((x \odot y) \oplus z) = (y \oplus z) \odot ((y \odot z) \oplus x)$,
(YY) $x \oplus y = x \implies x \odot (y \oplus z) = y \oplus (x \odot z)$.

(1') Dually, a *left-XY algebra* is an involutive left-m-MEL algebra $\mathcal{A} = (A, \odot, ^-, 1)$ verifying the additional axioms: for all $x, y, z \in A$,
(XXd) $(x \odot y) \oplus ((x \oplus y) \odot z) = (y \odot z) \oplus ((y \oplus z) \odot x)$,
(YYd) $x \odot y = x \implies x \oplus (y \odot z) = y \odot (x \oplus z)$.

Note that:

- the property (XX) appears, in [1] (see Lemma 6.3), as $\sigma_1(x, y, z) = \sigma_4(x, y, z)$, and the property (XXd) appears (with a permutation) in [5], Prop. 1.6.2, as property (1.16) verified by any MV algebra;
- the property (YY) appears in [1] as Lemma 6.9.

Denote by $\mathbf{XY}^R$ the class of all right-XY algebras and by $\mathbf{XY}$ the class of all left-XY algebras. Note that these classes are not varieties.

We shall say that a XY algebra is *proper*, if it is not an MV algebra.

Note that a right-XY algebra does not verify the properties (m-Re^R), (m-An^R), (m-Tr^R) (hence, the binary relation $\leq_m$ is not reflexive, antisymmetric or transitive) and (m-B^R).

Here are some properties:

Corollary 2.17 ([15], Cor. 2.2.51) *Let* $\mathcal{A} = (A, \oplus, ^-, 0)$ *be a right-XY algebra. Then,*

$$(\mathrm{XX}) \iff (\mathrm{XXd}).$$

Proposition 2.18 ([15], Prop. 2.2.52)

$$(\mathrm{DN}) \implies ((\mathrm{YY}) \Leftrightarrow (\mathrm{YYd})).$$

Proposition 2.19 ([15], Prop. 2.2.53) *If $\mathcal{A} = (A, \oplus, ^-, 0)$ is a right-XY algebra, then $(A, \odot, ^-, 1)$ is a left-XY algebra, where $x \odot y \overset{def.}{=} (x^- \oplus y^-)^-$, $1 \overset{def.}{=} 0^-$.*

Proposition 2.20 ([15], Prop. 2.2.54) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-XY algebra. For all $x, y \in A$,*

$$x \oplus y = x \iff x \odot y = y.$$

Proposition 2.21 ([15], Prop. 2.2.56) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-XY algebra. For all $x \in A$,*

$$0 \leq_m x \leq_m 1 \quad \textit{and} \quad 0 \leq_m^M x.$$

2.3.1 Cancellative XY Algebras

An even better 'ancestor' (generalization) of MV algebras are those right-XY algebras verifying the cancellative property (C) (verified by any MV algebra).

Definition 2.22 ([15], Def. 2.2.16) (The dual one is omitted)
A right-XY algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ is cancellative, if the following cancellative selfdual property (C) *holds: for all $x, y, z \in A$,*

(C) $z \oplus x = z \oplus y$ *and* $z \odot x = z \odot y$ *imply* $x = y$.

Denote by $\mathbf{CXY}^R$ the class of cancellative right-XY algebras. Hence, we have:

$$\mathbf{CXY}^R \subset \mathbf{XY}^R, \quad \textit{namely}$$
$$\mathbf{CXY}^R = \mathbf{XY}^R + (\mathrm{C}).$$

2.3.2 The Property (Z) and the Z Algebras

Consider the following property:

(Z) $(x \oplus y) \oplus (x \odot y) = x \oplus y$ and, dually,
(Zd) $(x \odot y) \odot (x \oplus y) = x \odot y$.

Note that the property (Z) appears as the property (1.14) verified by any (right-) MV algebra, in [5], Prop. 1.6.2, and appears in [1] (its dual also), in the proof of Lemma 6.10.

Related to the property (Z), we shall introduce a new particular involutive m-MEL algebra, called *Z algebra*, as follows.

Definitions 2.23 ([15], Defs. 2.2.17)

(1) A *right-Z algebra* is an involutive right-m-MEL algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ verifying the additional axiom (Z).

(1') Dually, a *left-Z algebra* is an involutive left-m-MEL algebra $\mathcal{A} = (A, \odot, ^-, 1)$ verifying the additional axiom (Zd).

Denote by $\mathbf{Z}^R$ the class of all right-Z algebras and by $\mathbf{Z}$ the class of all left-Z algebras. Note that these classes are varieties. We have:

$$\mathbf{Z}^R = \mathbf{m\text{-}MEL}^R_{(\mathrm{DN})} + (\mathrm{Z}).$$

We shall say that a Z algebra is *proper*, if it is not a XY algebra or an MV algebra.

Here are some properties:

Proposition 2.24 ([15], Prop. 2.2.18) *We have:*

$$(\mathrm{DN}) \implies ((\mathrm{Z}) \Leftrightarrow (\mathrm{Zd})).$$

Proposition 2.25 ([15], Prop. 2.2.19) *If $\mathcal{A} = (A, \oplus, ^-, 0)$ is a right-Z algebra, then $(A, \odot, ^-, 1)$ is a left-Z algebra, where $x \odot y \overset{def.}{=} (x^- \oplus y^-)^-$, $1 \overset{def.}{=} 0^-$.*

Proposition 2.26 ([15], Prop. 2.2.36) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an involutive right-m-MEL algebra. Then,*

$$(\mathrm{XX}) \implies (\mathrm{Z}).$$

Corollary 2.27 ([15], Cor. 2.2.37) *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be a right-XY algebra. Then,* (Z) *and* (Zd) *hold, i.e. $\mathcal{A}$ is a right-Z algebra.*

2.3.3 Connections Between Z Algebras, XY Algebras and MV Algebras

We know now that any XY algebra is an Z algebra (Corollary 2.27) and that any Z algebra verifying (XX) and (YY) is an XY algebra. We write:

$$\mathbf{XY}^R \subseteq \mathbf{Z}^R, \qquad \mathbf{Z}^R + (\mathrm{XX}) + (\mathrm{YY}) = \mathbf{XY}^R.$$

The goal of this subsubsection, reached in Corollary 2.30, is to prove that the XY algebra is an ancestor of the MV algebra.

- First, we shall prove, in Theorem 2.28 below, that any MV algebra is an XY algebra.

  Indeed, any right-MV algebra is an involutive right-m-MEL algebra, by definition, and it verifies the properties (Z) ([15], Prop. 2.2.59), (XX) ([15], Prop. 2.2.60), (YY) ([15], Prop. 2.2.62). Hence, we have:

  Theorem 2.28 ([15], Th. 2.2.63) *Any right-MV algebra is a cancellative right-XY algebra.*

  We write:
  $$\mathbf{MV}^R \subseteq \mathbf{CXY}^R.$$

- Second, we shall prove, in Corollary 2.30 below, that right-XY algebras verifying (m-Re^R) coincide with right-MV algebras.

  Corollary 2.29 *([15], Cor. 2.2.66)*

  In any involutive right-m-MEL algebra, we have:
  $$(\mathrm{XX}) + (\mathrm{m-Re^R}) \Longleftrightarrow (\vee_{\mathrm{m}} - \mathrm{comm}).$$

  Corollary 2.30 ([15], Cor. 2.2.68) *In any right-XY algebra, we have:*
  $$(\mathrm{m-Re^R}) \Longleftrightarrow (\vee_{\mathrm{m}} - \mathrm{comm}).$$

  Note that the above Corollary 2.30 says that right-XY algebras verifying (m-Re^R) coincide with right-MV algebras, i.e.
  $$\mathbf{XY}^R + (\mathrm{m-Re^R}) = \mathbf{MV}^R.$$

  Thus, we have the connections from the following Figure 1 ([15], Fig. 2.3).

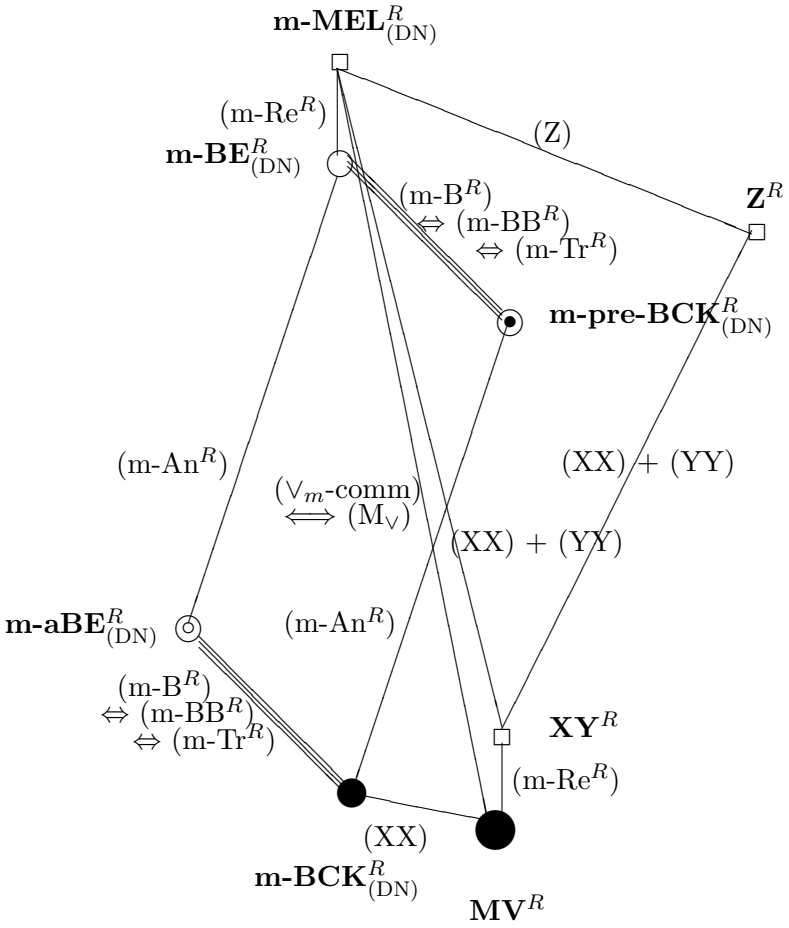


Figure 1: The connections between $\mathbf{m\text{-}MEL}^R_{(\mathrm{DN})}$, $\mathbf{m\text{-}BCK}^R_{(\mathrm{DN})}$, $\mathbf{Z}^R$, $\mathbf{XY}^R$ and $\mathbf{MV}^R$

2.3.4 Totally Ordered Proper XY Algebras Do Not Exist

We shall prove here that totally ordered proper XY algebras do not exist: they are MV algebras.

Note that, if a right-XY algebra $\mathcal{A}$ is *totally ordered by* $\leq_m$, then $x \oplus x^- = 1$, for all $x \in A$, i.e. (m-Re^R) holds. Hence, we immediately obtain, by Corollary 2.30, the following result.

Proposition 2.31 ([15], Prop. 2.2.57) *If a right-XY algebra $\mathcal{A}$ is totally ordered by $\leq_m$, then it is an MV algebra.*

By Proposition 2.10, we obtain immediately the following result.

Corollary 2.32 ([15], Cor. 2.2.58) *If a right-XY algebra $\mathcal{A}$ is totally ordered by $\leq_m^M$, then it is an MV algebra.*

3 Strong m_0-ME Structures. Strong m_0-ME Positive Cones

In this section, we introduce and study, in the new framework centered on m-BCK algebras, the strong m_0-ME structure and the strong m_0-ME positive cone as the 'ancestors' of commutative lattice-ordered group with strong unit and its positive cone, respectively.

3.1 Involutive m_0-ME Algebras

Recall the following definitions from [13], Chapter 7:

Definitions 3.1 ([15], Defs. 3.1.6) (See [13], subsection 7.1.1)

(1) A m_0-ME algebra is an algebra $\mathcal{M} = (M, +, -, 0_M)$ of type $(2, 1, 0)$ verifying the following axioms: for all $x, y, z \in M$,

(SU) $x + 0_M = x = 0_M + x$,
(Scomm) $x + y = y + x$,
(Sass) $(x + y) + z = x + (y + z)$,
(N0) $-0_M = 0_M$,

i.e. it is a commutative additive monoid verifying (N0).

(1') Dually, a m_1-ME algebra is an algebra $\mathcal{M} = (M, \cdot, ^{-1}, 1_M)$ of type $(2, 1, 0)$ verifying the following axioms: for all $x, y, z \in M$,

(PU) $x \cdot 1_M = x = 1_M \cdot x$,
(Pcomm) $x \cdot y = y \cdot x$,
(Pass) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(N1) ${1_M}^{-1} = 1_M$,
i.e. it is a commutative multiplicative monoid verifying (N1).

Denote by m_0**-ME** the class of all m_0-ME algebras and by m_1**-ME** the class of all m_1-ME algebras.

Any commutative group is a m_0-ME algebra ([15], Rem. 3.1.7).

Define, in a m_0-ME algebra $\mathcal{M}$, a basic internal binary relation (i.e. defined by using the operations on M) (see [13]) $\leq_{m_0}$ by: for all $x, y \in M$,

$$x \leq_{m_0} y \overset{def.}{\Longleftrightarrow} y + (-x) = 0_M.$$

We shall denote:

- the *reflexivity* of $\leq_{m_0}$, by (m_0-Re),
- the *antisymmetry* of $\leq_{m_0}$, by (m_0-An) and
- the *transitivity* of $\leq_{m_0}$, by (m_0-Tr),

hence we have the following properties:

(m_0-Re) $x + (-x) = 0_M$,
(m_0-Re') $x \leq_{m_0} x$;

(m_0-An) $y + (-x) = 0_M$ and $x + (-y) = 0_M$ imply $x = y$,
(m_0-An') $x \leq_{m_0} y$ and $y \leq_{m_0} x$ imply $x = y$;

(m_0-Tr) $y + (-x) = 0_M$ and $z + (-y) = 0_M$ imply $z + (-x) = 0_M$,
(m_0-Tr') $x \leq_{m_0} y$ and $y \leq_{m_0} z$ imply $x \leq_m z$.

Any commutative group is a m_0-ME algebra verifying (m_0-Re) ([15], Rem. 3.1.8).

Definitions 3.2 ([15], Defs. 3.1.9) (1) An *involutive* m_0*-ME algebra* is a m_0-ME algebra $\mathcal{M} = (M, +, -, 0_M)$ verifying the following axiom: for all $x \in M$,

(DN) $-(-x) = x$, or $= x = x$.

(1') Dually, an *involutive m_1-ME algebra* is a m_1-ME algebra $\mathcal{M} = (M, \cdot, ^{-1}, 1_M)$ verifying the following axiom: for all $x \in M$,

(DN) $\quad (x^{-1})^{-1} = x$.

Denote by m_0-$\mathbf{ME}_{(\mathrm{DN})}$ the class of all involutive m_0-ME algebras and by m_1-$\mathbf{ME}_{(\mathrm{DN})}$ the class of all involutive m_1-ME algebras.

Any commutative group is an involutive m_0-ME algebra verifying (m_0-Re) ([15], Remark 3.1.10).

Let $\mathcal{M} = (M, +, -, 0_M)$ be an involutive m_0-ME algebra. Let us introduce a new operation, $\cdot$, defined by: for all $x, y \in M$,

$$x \cdot y \overset{def.}{=} -((-x) + (-y)). \tag{5}$$

Then, we have the following result.

Proposition 3.3 ([15], Prop. 3.1.11) *Let $\mathcal{M} = (M, +, -, 0_M)$ be an involutive m_0-ME algebra. Then, the algebra $(M, \cdot, ^{-1} \overset{def.}{=} -, 1_M \overset{def.}{=} 0_M)$ is an involutive m_1-ME algebra.*

Proposition 3.4 ([15], Prop. 3.1.12) (See ([13], Rems. 7.1.1 (4))) Any (involutive) m_0-ME algebra verifying (m_0-Re) (i.e. $x + (-x) = 0_M$) is a commutative group.

Hence, the (involutive) m_0-ME algebras verifying (m_0-Re) coincide with the commutative groups. We write:

$$\mathbf{m_0 - ME}_{(\mathrm{DN})} \ + \ (\mathrm{m}_0 - \mathrm{Re}) \ = \ \mathbf{c.group}.$$

We introduce now the following new notions.

Definitions 3.5 ([15], Defs. 3.2.5)

(1) A *m_0-ME structure* is a structure $\mathcal{M} = (M, \leq_m, +, \smile, 0_M)$ such that:

(i) $(M, +, \smile, 0_M)$ is an involutive m_0-ME algebra,

(ii) $\leq_m$ is an external binary relation on M (i.e. not defined by the operations on M) such that $0_M \leq_m 0_M$ ('m' comes from 'magma').

(1') Dually, a m_1-*ME structure* is a structure $\mathcal{M} = (M, \leq_m, \cdot, ^{-1}, 1_M)$ such that:

(i') $(M, \cdot, ^{-1}, 1_M)$ is an involutive m_1-ME algebra,

(ii') $\leq_m$ is an external binary relation on M such that $1_M \leq_m 1_M$ ('m' comes from 'magma').

Note that we shall freely write $x \leq_m y$ or $y \geq_m x$ in the sequel.

3.2 Strong m_0-ME Structures

Definition 3.6 ([15], Defs. 3.2.6) (The dual one is omitted) *A strong m_0-ME structure is a structure*

$$\mathcal{M}_u = (M, \leq_m, +, \smile, 0_M, u = u_M)$$

such that:

(i) $(M, \leq_m, +, \smile, 0_M)$ *is a m_0-ME structure,*

(ii) u_M *is an element of M verifying:*

($\mathrm{s}0^R$) $0_M <_m u$;

($\mathrm{s}1^R$) *for each $x \in M$, there exists $n_x \in \mathbb{N}$ such that $x, \smile x \leq_m n_x u$;*

($\mathrm{s}2^R$)

(i) for each $x \geq_m 0_M$, there exists $n_x \in \mathbb{N}$, and there exist the unique elements of M:

$$0_M \leq_m x_1, x_2, \ldots, x_{n_x} \leq_m u$$

(hence $0_M \leq_m \smile x_1, \smile x_2, \ldots, \smile x_{n_x} \leq_m u$),
s.t. $x = x_1 + \ldots + x_{n_x}$ $(\leq_m n_x u)$,
$\smile x = \smile x_1 \smile \ldots \smile x_{n_x}$ $(\leq_m n_x u)$,

(ii) if $x = x_1 + \ldots + x_n \geq_m 0_M$ and $y = y_1 + \ldots + y_n \geq_m 0_M$, then $x = y \iff x_i = y_i$, $i = 1, \ldots, n$,

(iii) if $x = x_1 + \ldots + x_n \geq_m 0_M$ and $y = y_1 + \ldots + y_n \geq_m 0_M$, then $x \leq_m y \iff x_i \leq_m y_i$, $i = 1, \ldots, n$;

($\mathrm{s}3^R$) *for $0_M \leq_m x, y \leq_m u$ (i.e. $x, y \in [0_M, u]$),*

(i) there exist $\inf_m(u, x + y)$, $\smile \inf_m(u, \smile x \smile y)$ and belong to $[0_M, u]$,

(ii) $\inf_m(u, x+u) = u, \quad \smile \inf_m(u, \smile x \smile 0_M) = 0_M,$
$\smile \inf_m(u, \smile x \smile u) = x,$

(iii) $x \leq_m y \ (x, y \neq 0_M) \Longleftrightarrow u \leq_m y - x;$

($s4^R$) *for* $0_M \leq_m x, y, z \leq_m u,$

(i) there exist and are equal
$\inf_m(u, x + \inf_m(u, y+z)) = \inf_m(u, x + (y+z)),$

(ii) $\inf_m(u, \smile \inf_m(u, \smile x \smile y)) = \smile \inf_m(u, \smile x \smile y);$

($s5^R$) *for* $0_M \leq_m x_0, x_1, \ldots, x_n, y_0 \leq_m u,$

(V^{00}) $x_0 + y_0 = \inf_m(u, x_0 + y_0) \smile \inf_m(u, \smile x_0 \smile y_0),$

(V^{10}) *if* $\inf_m(u, x_0 + x_1) = x_0$, *then* $x_0 + x_1 + y_0 = \inf_m(u, x_0 + y_0)$
$\smile \inf_m(u, \smile x_0 \smile \inf_m(u, x_1 + y_0)) \smile \inf_m(u, \smile x_1 \smile y_0),$

(V^{n0}) *if* $\inf_m(u, x_i + x_{i+1}) = x_i$, *for* $i = 0, 1, \ldots, n-1$, *then*
$\sum_{i=0}^{n} x_i + y_0 = \sum_{i=0}^{n+1} c_i$, *where*
$c_i = \smile \inf_m(u, \smile x_{i-1} \smile \inf_m(u, x_i + y_0))$, *with* $x_{-1} = u$, $x_{n+1} = 0_M$;

(XX_u^R) *for* $0_M \leq_m x, y, z \leq_m u$, *there exist and are equal*
$\smile \inf_m(u, \ \smile \inf_m(u, x+y) \smile \inf_m(u, \ z \smile \inf_m(u, \smile x \smile y)))$
$= \smile \inf_m(u, \ \smile \inf_m(u, y+z) \smile \inf_m(u, \ x \smile \inf_m(u, \smile y \smile z)));$

(YY_u^R) *for* $0_M \leq_m x, y, z \leq_m u$,
if $\inf_m(u, x+y) = x$, *then there exist and are equal*
$\smile \inf_m(u, \smile x \smile \inf_m(u, y+z)) = \inf_m(u, y \smile \inf_m(u, \smile x \smile z));$

(C_u^{1R}) $x + u = y + u \implies x = y$, *for all* $x, y \in M;$

(C_{pu}^{1R}) $x \leq_m y \iff x + u \leq_m y + u$, *for all* $x, y \in M;$

(m-Tr_u^R) $x \leq_m nu$ *and* $nu \leq_m ru \implies x \leq_m ru$, *for all* $x \in M$, $n, r \in \mathbb{N};$

(Nu^{1R}) $\smile u = u;$

(NNu^{1R}) $\smile (\smile u) = u;$

(S^{1R}) $\smile (x+u) = \smile x + u$, *for each* $x \in M$,
where $\smile x \smile y$ *means* $\smile x + \smile y$.

3.3 Strong m_0-ME Positive Cones

Definition 3.7 ([15], Defs. 3.3.1) (The dual one is omitted)
A strong m_0-ME positive cone is a structure

$$\mathcal{M}_u^+ = (M^+, \leq_m, +, \smile, 0_{M^+}, u = u_{M^+})$$

such that:

(i) $(M^+, \leq_m, +, \smile, 0_{M^+})$ *is a m_0-ME structure,*

(ii) $0_{M^+} <_m u$ *is an element of M^+ verifying:*
$(\mathrm{s}0^R)$ $0_{M^+} \leq_m x$, *for each $x \in M^+$;*
$(\mathrm{s}1^R)$ *for all $x \in M^+$, there exists $n_x \in \mathbb{N}$ such that $x, \smile x \leq_m n_x u$;*
$(\mathrm{s}2^{+R})$

(i) for all $x \in M^+$, there exists $n_x \in \mathbb{N}$ and there exist the unique elements of M^+: $x_1, x_2, \ldots, x_{n_x} \leq_m u$ *(hence* $0_{M^+} \leq_m \smile x_1, \smile x_2, \ldots, \smile x_{n_x} \leq_m u$*) s.t.* $x = x_1 + \ldots + x_{n_x}$ $(\leq_m n_x u)$, $\smile x = \smile x_1 \smile \ldots \smile x_{n_x}$ $(\leq_m n_x u)$,

(ii) if $x = x_1 + \ldots + x_n$ and $y = y_1 + \ldots + y_n$, then $x = y \iff x_i = y_i$, $i = 1, \ldots, n$,

(iii) if $x = x_1 + \ldots + x_n$ and $y = y_1 + \ldots + y_n$, then $x \leq_m y \iff x_i \leq_m y_i$, $i = 1, \ldots, n$;

$(\mathrm{s}3^R)$ *for $0_{M^+} \leq_m x, y \leq_m u = u_{M^+}$ (i.e. $x, y \in [0_{M^+}, u]$),*

(i) there exist $\inf_m(u, x + y)$, $\smile \inf_m(u, \smile x \smile y)$ *and belong to* $[0_{M^+}, u]$,

(ii) $\inf_m(u, x + u) = u$, $\smile \inf_m(u, \smile x \smile 0_{M^+}) = 0_{M^+}$, $\smile \inf_m(u, \smile x \smile u) = x$,

(iii) $x \leq_m y$ $(x, y \neq 0_{M^+}) \iff u \leq_m y \smile x$;

$(\mathrm{s}4^R)$ *for $0_{M^+} \leq_m x, y, z \leq_m u = u_{M^+}$,*

(i) there exist and are equal $\inf_m(u, x + \inf_m(u, y + z)) = \inf_m(u, x + (y + z))$,

(ii) $\inf_m(u, \smile \inf_m(u, \smile x \smile y)) = \smile \inf_m(u, \smile x \smile y)$;

($s5^R$) *for* $0_{M^+} \leq_m x_0, x_1, \ldots, x_n, y_0 \leq_m u = u_{M^+}$,

(V^{00}) $x_0 + y_0 = \inf_m(u, x_0 + y_0) \smile \inf_m(u, \smile x_0 \smile y_0)$,

(V^{10}) *if* $\inf_m(u, x_0 + x_1) = x_0$, *then* $x_0 + x_1 + y_0 = \inf_m(u, x_0 + y_0) \smile \inf_m(u, \smile x_0 \smile \inf_m(u, x_1 + y_0)) \smile \inf_m(u, \smile x_1 \smile y_0)$,

(V^{n0}) *if* $\inf_m(u, x_i + x_{i+1}) = x_i$, *for* $i = 0, \ldots, n-1$, *then* $\sum_{i=0}^{n} x_i + y_0 = \sum_{i=0}^{n+1} c_i$, *where* $c_i = \smile \inf_m(u, \smile x_{i-1} \smile \inf_m(u, x_i + y_0))$, *with* $x_{-1} = u$, $x_{n+1} = 0_{M^+}$;

(XX_u^R) *for* $0_{M^+} \leq_m x, y, z \leq_m u = u_{M^+}$, *there exist and are equal* $\smile \inf_m(u, \smile \inf_m(u, x + y) \smile \inf_m(u, z \smile \inf_m(u, \smile x \smile y)))$ $= \smile \inf_m(u, \smile \inf_m(u, y + z) \smile \inf_m(u, x \smile \inf_m(u, \smile y \smile z)))$;

(YY_u^R) *for* $0_{M^+} \leq_m x, y, z \leq_m u = u_{M^+}$, *if* $\inf_m(u, x + y) = x$, *then there exist and are equal* $\smile \inf_m(u, \smile x \smile \inf_m(u, y + z)) = \inf_m(u, y \smile \inf_m(u, \smile x \smile z))$;

(C_u^{1R}) $x + u = y + u \Longrightarrow x = y$, *for all* $x, y \in M^+$;

(C_{pu}^{1R}) $x \leq_m y \iff x + u \leq_m y + u$, *for all* $x, y \in M^+$;

($m\text{-}Tr_u^R$) $x \leq_m nu$ *and* $nu \leq_m ru \Longrightarrow x \leq_m ru$, *for each* $x \in M^+$, $n, r \in \mathbb{N}$;

(Nu^{+1R}) $\smile u = 0_{M^+}$;

(NNu^{1R}) $\smile (\smile u) = u$;

(S^{+1R}) $\smile (x + u) = \smile x \smile u = \smile x$, *for each* $x \in M^+$, *where* $\smile x \smile y$ *means* $\smile x + \smile y$.

Remark 3.8 ([15], Remark 3.3.13) (The dual one is omitted)
The properties ($s2^{+R}$), (Nu^{+1R}) and (S^{+1R}) of a strong m_0-ME positive cone differ from the properties ($s2^R$), (Nu^{1R}) and (S^{1R}), respectively, of a strong m_0-ME structure.

We show in the book the connections existing between the strong m_0-ME structure and the commutative l-group with strong unit and between the strong m_0-ME positive cone and the positive-unital commutative l-group.

4 The Two Steps and the Four Functors

In the book, we generalize Mundici's equivalence analogously as Abbadini did in [1], i.e. in two steps, with four functors.

4.1 The Step 1: The Functors $\mathbf{c}_\mathbf{m}^+$ and $\mathbf{T}_\mathbf{m}$

The algebraic structures involved in the first step are the *strong m_0-ME structures* (Definition 3.6) and the *strong m_0-ME positive cones* (Definition 3.7), which both are involutive m_0-ME algebras verifying some properties.

We denote with $\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME})$ the category of strong m_0-ME structures with homomorphisms.

We denote with $\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME}^+)$ the category of strong m_0-ME positive cones with homomorphisms.

4.1.1 The Functor $\mathbf{c}_\mathbf{m}^+$

Let $\mathcal{M}_u = (M, \leq_m, +, \smile, 0_M, u_M)$ be a strong m_0-ME structure and let define the positive cone as follows:

$$M^+ \stackrel{def.}{=} \{x \in M \mid x \geq_m 0_M, \textit{ such that } (Nu^{+1R}),\ (S^{+1R}) \textit{ hold}\}.$$

If we denote with the same symbols $+$ and $\smile$ the restrictions on M^+ of the operations $+$ and $\smile$ on M and if we denote with $\leq_m$ the restriction on M^+ of the binary relation $\leq_m$ on M, then we obtain the structure

$$\mathcal{M}_u^+ = (M^+, \leq_m, +, \smile, 0_{M^+}, u_{M^+}),$$

where $0_{M^+} = 0_M$ and $u_{M^+} = u_M$, which, obviously, is a *strong m_0-ME positive cone*.

We put:

$$\mathbf{c}_\mathbf{m}^+(\mathcal{M}_u) = \mathcal{M}_u^+.$$

Given a morphism $f : M \longrightarrow N$ of strong m_0-ME structures $\mathcal{M}_u = (M, \leq_m, +, \smile, 0_M, u = u_M)$ and $\mathcal{N}_v = (N, \leq_m, +, \smile, 0_N, v = u_N)$, f restricts to a function f^+ from M^+ to N^+ and, obviously, f^+ is a morphism of strong m_0-ME positive cones $\mathcal{M}_u^+ = (M^+, \leq_m, +, \smile, 0_{M^+}, u = u_{M^+})$ and $\mathcal{N}_v^+ = (N^+, \leq_m, +, \smile, 0_{N^+}, v = u_{N^+})$.

This establishes that $\mathbf{c}_\mathbf{m}^+$ is a functor $\mathbf{c}_\mathbf{m}^+ : \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME}) \longrightarrow \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME}^+)$ that maps $\mathcal{M}_u$ to $\mathcal{M}_u^+$ and maps a morphism $f : M \longrightarrow N$ to its restriction

$$f^+ = \mathbf{c}_\mathbf{m}^+(f) : \mathbf{c}_\mathbf{m}^+(M) = M^+ \longrightarrow \mathbf{c}_\mathbf{m}^+(N) = N^+.$$

4.1.2 The Functor $\mathbf{T_m}$

The name '$\mathbf{T_m}$' comes here from 'turning/tipping' (versus 'translations', from [1]), because the entire construction made by this functor is a turning/tipping point (or an inflexion point): it differs essentially from the construction from [5], because **the strong $\mathbf{m_0}$-ME positive cone does not verify the cancellation property (C)**; it is analogous to that from [1].

Let $\mathcal{M}_u^+ = (M^+, \leq_m, +, \smile, 0_{M^+}, u = u_{M^+})$ be a strong m_0-ME positive cone, **which is not cancellative as happens in [5]**.

Denote $M^+ \times \mathbb{N} = \{(x, n) \mid x \in M^+, n \in \mathbb{N}\}$. We consider the binary relation $\sim$ defined on $M^+ \times \mathbb{N}$ as follows:

$$(x, n) \sim (y, p) \overset{def.}{\iff} x + pu_{M^+} = y + nu_{M^+}.$$

We shall denote the equivalence class of $(x, n) \in M^+ \times \mathbb{N}$ by $\overbrace{(x, n)}$. The set of all equivalence classes (the quotient set) is denoted by $\mathbf{T_m}(M^+)$:

$$\mathbf{T_m}(M^+) := \frac{M^+ \times \mathbb{N}}{\sim}$$

and is endowed with the operations of a strong m_0-ME structure:

$$\mathbf{0} \overset{def.}{=} \overbrace{(0_{M^+}, 0)} \quad ([15],\ (5.2)),$$

$$\mathbf{u} \overset{def.}{=} \overbrace{(u_{M^+}, 0)} \quad ([15],\ (5.3)),$$

$$\smile \overbrace{(x + mu_{M^+}, n)} \overset{def.}{=} \overbrace{(\smile x + mu_{M^+}, n)} \quad ([15],\ (5.5)),$$

$$\overbrace{(X, n)} + \overbrace{(Y, p)} \overset{def.}{=} \overbrace{(X + Y, n + p)} \quad ([15],\ (5.4)),$$

$$\overbrace{(X, n)} \preceq_m \overbrace{(Y, p)} \overset{def.}{\iff} X + pu_{M^+} \leq_m Y + nu_{M^+} \quad ([15],\ (5.6)).$$

We define the infimum related to $\preceq_m$, $\inf_m$, by:

$$\inf_m(\overbrace{(X, n)}, \overbrace{(Y, p)}) \overset{def.}{=} \overbrace{(\inf_m(X + pu_{M^+}, Y + nu_{M^+}), n + p)} \quad ([15],\ (5.7)).$$

Then, $(\mathbf{T_m}(M^+) = \frac{M^+ \times \mathbb{N}}{\sim}, \preceq_m, +, \smile, \mathbf{0}, \mathbf{u})$ is a strong m_0-ME structure ([15], Th. 5.1.27). $\mathbf{T_m}$ is a functor ([15], Prop. 5.2.1).

4.1.3 The Connections Between the Functors $\mathbf{c}_{\mathbf{m}}^{+}$ and $\mathbf{T}_{\mathbf{m}}$

The functors $\mathbf{c}_{\mathbf{m}}^{+}$ and $\mathbf{T}_{\mathbf{m}}$ are not quasi-inverses, thus the categories $\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME})$ and $\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME}^{+})$ are not equivalent (the functor $\mathbf{c}_{\mathbf{m}}^{+}$ is not an equivalence) ([15], Cor. 6.3.3).

4.2 The Step 2: The Functors $\mathbf{G}_{\mathbf{m}}^{+}$ and $\mathbf{U}_{\mathbf{m}}$

The algebraic structures involved in the second step are the above *strong m_0-ME positive cones* and the *right-XY algebras* (Definition 2.16).

We denote with $\mathcal{C}(\mathbf{XY}^{R})$ the category of right-XY algebras with homomorphisms.

4.2.1 The Functor $\mathbf{G}_{\mathbf{m}}^{+}$

Let $\mathcal{A} = (A, \oplus, {}^{-}, 0 = 0_A)$ be a right-XY algebra, with $1 = 1_A \overset{def.}{=} 0^{-}$.

(1) A *good pair* of A is a pair (x, y) of elements of A such that $x \oplus y = x$.

(2) A *good sequence* of A is a sequence $\mathbf{a} = (a_0, a_1, \ldots)$ of elements of A such that:

   (i) for each $i = 0, 1, \ldots$, the pair (a_i, a_{i+1}) is good and

   (ii) there is $n \in \mathbb{N}$, such that $a_k = 0$ for all $k > n$, i.e. $\mathbf{a} = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$. We shall simply write: $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ ([15], Defs. 7.1.1).

Note that, like in MV algebras, (x, y) is a good pair if and only if $x \odot y = y$.

We shall denote by $G^{+}(A)$ the set of good sequences of $\mathcal{A}$ ('G' comes from 'good'). We endow $G^{+}(A)$ with the structure of a strong $\mathrm{m_0}$-ME positive cone, as follows.

We let (x) denote the good sequence $(x, 0, 0, \ldots)$, hence (0) denotes the good sequence $(0, 0, 0, \ldots)$ and (1) denotes the good sequence $(1, 0, 0, 0, \ldots)$. We put $0_{G^{+}(A)} = (0)$ and $u_{G^{+}(A)} = (1)$.

Given two good sequences $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ and $\mathbf{b} = (b_0, b_1, b_2, \ldots)$ of $\mathcal{A}$, like Abbadini in [1], we define their sum

$$\mathbf{a} + \mathbf{b} \overset{def.}{=} (c_0, c_1, c_2, \ldots),$$

where:

$$c_n \overset{def.}{=} (a_0 \oplus b_n) \odot (a_1 \oplus b_{n-1}) \odot \ldots \odot (a_{n-1} \oplus b_1) \odot (a_n \oplus b_0)$$

or, equivalently,

$$c_n \overset{def.}{=} b_n \oplus (a_0 \odot b_{n-1}) \oplus (a_1 \odot b_{n-2}) \oplus \ldots \oplus (a_{n-2} \odot b_1) \oplus (a_{n-1} \odot b_0) \oplus a_n.$$

Given the good sequence $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}, a_n)$ of $\mathcal{A}$, we define:

$\smile \mathbf{a} = \smile (a_0, a_1, \ldots, a_{n-1}, a_n) \overset{def.}{=} (a_n^-, a_{n-1}^-, \ldots, a_1^-, a_0^-)$ ([15], (7.10)).

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}, a_n)$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1}, b_n)$ be two good sequences of $\mathcal{A}$. We define on $G^+(A)$ the external binary relation (i.e. not defined by the operations on $G^+(A)$) $\leq_m$, by:

$\mathbf{a} \leq_m \mathbf{b} \overset{def.}{\Longleftrightarrow} a_i \leq_m b_i$, *for* $i = 0, 1, \ldots, n$, *in* $\mathcal{A}$.

For the good sequences $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}, a_n)$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1}, b_n)$, and the binary relation $\leq_m$ on $G^+(A)$, we set:

$\inf_m(\mathbf{a}, \mathbf{b}) \overset{def.}{=} (\inf_m(a_0, b_0), \inf_m(a_1, b_1), \ldots, \inf_m(a_n, b_n))$ ([15], (7.13)),

$\sup_m(\mathbf{a}, \mathbf{b}) \overset{def.}{=} (\sup_m(a_0, b_0), \sup_m(a_1, b_1), \ldots, \sup_m(a_n, b_n))$ ([15], (7.14)),

and they exist, if $\inf_m(a_0, b_0), \inf_m(a_1, b_1), \ldots, \inf_m(a_n, b_n)$ and $\sup_m(a_0, b_0), \sup_m(a_1, b_1), \ldots, \sup_m(a_n, b_n)$, respectively, do exist. When they exist, the problem is if they are good sequences.

For every right-XY algebra $\mathcal{A}$, the structure

$$(G^+(A), \leq_m, +, \smile, 0_{G^+(A)}, u_{G^+(A)})$$

is a strong m_0-ME positive cone ([15], Th. 7.1.77).

We put $\mathbf{G}_\mathbf{m}^+(\mathcal{A}) = (G^+(A), \leq_m, +, \smile, 0_{G^+(A)}, u_{G^+(A)})$. $\mathbf{G}_\mathbf{m}^+$ is a functor ([15], Prop. 7.2.1).

Open problems 4.1 ([15], Op. probs. 7.1.45)

1. *Find an example of cancellative proper XY algebra* $\mathcal{A}$ *whose* $G^+(A)$ *verifies the cancellative property* (C).

2. *Find an example of cancellative XY algebra* $\mathcal{A}$ *whose* $G^+(A)$ *does not verify the property* (C) *for some* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *of* $G^+(A)$ *or prove that the commutative monoid* $G^+(A)$ *of any cancellative XY algebra* $\mathcal{A}$ *verifies the cancellative property* (C).

Remark 4.2 ([15], Rem. 7.1.78) (See Open problems 4.1)

If there is at least one example of cancellative proper XY algebra $\mathcal{A}$ whose $G^+(A)$ verifies the cancellative property (C), then its strong m_0-ME positive cone $(G^+(A), \leq_m, +, \smile, 0_{G^+(A)}, u_{G^+(A)})$ verifies (C), and hence another construction of the functor $\mathbf{T_m}$ can be made, similar to that of $\mathbf{T}$ in MV algebras case.

4.2.2 The Unit Interval Functor $\mathbf{U_m}$

Let $\mathcal{M}_u^+ = (M^+, \leq_m, +, \smile, 0_{M^+}, u = u_{M^+})$ be a strong m_0-ME positive cone.

Let $[0_{M^+}, u_{M^+}] \stackrel{def.}{=} \{x \in M^+ \mid 0_{M^+} \leq_m x \leq_m u_{M^+}\} \subset M^+$ be the unit interval in $\mathcal{M}_u^+$. We endow $[0_{M^+}, u_{M^+}]$ with a structure of right-XY algebra, as follows.

For all $x, y \in [0_{M^+}, u_{M^+}]$, we define ([15], Defs. 8.1.1):

$x \oplus y \stackrel{def.}{=} \inf_m(u_{M^+}, x + y)$; $\neg x \stackrel{def.}{=} \smile x$, $x \neq 0_{M^+}$, $\neg 0_{M^+} \stackrel{def.}{=} u_{M^+}$.

Then, $([0_{M^+}, u_{M^+}], \oplus, \neg, 0_{M^+})$ is a right-XY algebra ([15], Th. 8.1.2).

We put: $\mathbf{U_m}(\mathcal{M}_u^+) = ([0_{M^+}, u_{M^+}], \oplus, \neg, 0_{M^+})$. $\mathbf{U_m}$ is a functor ([15], Prop. 8.2.1) ($\mathbf{U_m}$ stands for 'unit interval').

4.2.3 The Connections Between the Functors $\mathbf{G_m^+}$ and $\mathbf{U_m}$

The functors $\mathbf{U_m} : \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME^+}) \longrightarrow \mathcal{C}(\mathbf{XY}^R)$ and $\mathbf{G_m^+} : \mathcal{C}(\mathbf{XY}^R) \longrightarrow \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME^+})$ are quasi-inverses. Thus, the categories of strong m_0-ME positive cones and of right-XY algebras are equivalent ([15], Cor. 9.3.3).

4.3 Connections Between the Four Functors

We put $\mathbf{\Gamma_m} = \mathbf{U_m} \circ \mathbf{c_m^+}$ and $\mathbf{\Xi_m} = \mathbf{T_m} \circ \mathbf{G_m^+}$.

The main result of the book is the following ([15], Th. 10.1.1):

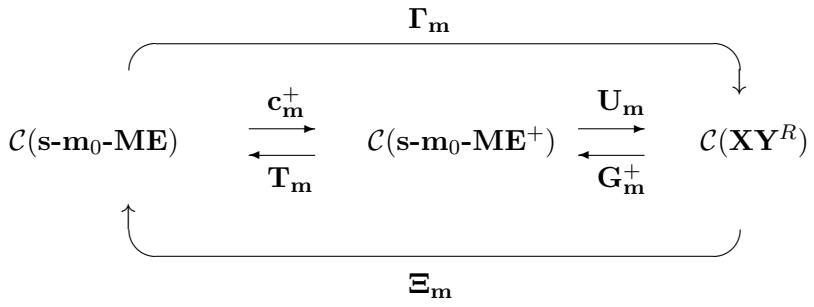
> The functor $\mathbf{\Gamma_m} = \mathbf{U_m} \circ \mathbf{c_m^+} : \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME}) \longrightarrow \mathcal{C}(\mathbf{XY}^R)$ is not an equivalence of categories, but the functor $\mathbf{U_m} : \mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME^+}) \longrightarrow \mathcal{C}(\mathbf{XY}^R)$ is an equivalence of categories.

Resuming, the two steps and the four functors are:

$$(M, \leq_m, +, \smile, 0_M, u_M) \quad \overset{\mathbf{c_m^+}}{\longrightarrow} \quad (M^+, \leq_m, +, \smile, 0_{M^+} = 0_M, u_{M^+} = u_M)$$
$$(\tfrac{M^+ \times \mathbb{N}}{\sim}, \preceq_m, +, \smile, \mathbf{0}, \mathbf{u}) \quad \overset{\mathbf{T_m}}{\longleftarrow} \quad (M^+, \leq_m, +, \smile, 0_{M^+}, u_{M^+})$$

$$(M^+, \leq_m, +, \smile, 0_{M^+}, u_{M^+}) \quad \overset{\mathbf{U_m}}{\longrightarrow} \quad ([0_{M^+}, u_{M^+}], \oplus, \neg, 0_{M^+})$$
$$(G^+(A), \leq_m, +, \smile, (0), (1)) \quad \overset{\mathbf{G_m^+}}{\longleftarrow} \quad (A, \oplus, ^-, 0)$$

and we have:



where 'm' comes from 'magma' and:

$\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME})$ is the category of strong m_0-ME structures,

$\mathcal{C}(\mathbf{s\text{-}m_0\text{-}ME^+})$ is the category of strong m_0-ME positive cones,

$\mathcal{C}(\mathbf{XY}^R)$ is the category of right-XY algebras.

We show how the original Mundici's equivalence is a particular case of our results (see Corollary 2.30 and also Connections from [15], Chapter 3 and [15], Appendix C).

5 Future Work

Resolve Open problems 4.1.

If (see Remark 4.2) there is at least an example of cancellative proper XY algebra $\mathcal{A}$ whose $G^+(A)$ verifies the cancellative property (C), then its strong m_0-ME positive cone $(G^+(A), \leq_m, +, \smile, 0_{G^+(A)}, u_{G^+(A)})$ verifies (C), and hence another construction of the functor $\mathbf{T_m}$ can be made, similar to that of $\mathbf{T}$ in MV algebras case. This other construction of $\mathbf{T_m}$ will lead almost surely to a categorical equivalence.

The right-pseudo-MV algebras were introduced in 1999 [8], [9], as non-commutative generalizations $(A, \oplus, ^-, ^\sim, 0, 1)$ of right-MV algebras $(A, \oplus, ^-, 0)$

and, in [6], a generalization of Mundici's equivalence to the non-commutative case is presented. Analogously with the old *non-commutative framework centered on pseudo-BCK algebras*, we have introduced and studied in 2024 [14] a new *non-commutative framework centered on m-pseudo-BCK algebras*, containing, among many new algebras as the m_0-pseudo-ME algebras, the m-pseudo-MEL algebras, the m-pseudo-BE algebras and the m-pseudo-BCK algebras ('m' comes from 'magma'), the groups and the pseudo-MV algebras; a group $(G, +, -, 0)$ coincides with a m_0-pseudo-ME algebra verifying the property (m_0-pRe) ([14], Th. 10.1.26) and a pseudo-MV algebra $(A, \oplus, ^-, ^\sim, 0, 1)$ is an involutive m-pseudo-MEL algebra verifying (prod) and the property ($\vee_m$-comm) ([14], Def. 15.3.15), where (prod) is $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$ and ($\vee_m$-comm) is $y \oplus (x^- \oplus y)^\sim = x \oplus (y^- \oplus x)^\sim$. Thus, an immediate further work could be to generalize the notions and the results from the book [15] to the non-commutative case, in the new non-commutative framework centered on m-pseudo-BCK algebras, and connect them to [6].

Acknowledgements
I thank the anonymous referee for his/her valuable comments which helped improve the quality of this paper.

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